Linear Stochastic Control Using the UDU^T Matrix Factorization

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The classical LQG stochastic control law is reformulated using the matrix factorization $S=UDU^T$. This method yields a statistical guidance analysis algorithm that is numerically superior to the classical solution yet requires negligible additional computation and storage. Moreover, experience with *U-D* algorithms has shown them to be adaptable and easy to implement on a variety of problems.

I. Introduction

THE classical solution to the discrete LQG stochastic control problem involves two matrix Ricatti equations – the covariance recursion associated with the Kalman filter, and the dual to this recursion required for the control gain calculation.[1] It is by now well established that the Kalman covariance recursions are numerically unreliable and can yield erroneous results.[2-4] By duality one should, therefore, also suspect the numerical integrity of the control matrix recursion. Moreover, conventional guidance analysis methods involve additional matrix calculations that are susceptible to numerical errors.

Computational problems inherent in the conventional guidance analysis algorithm can be avoided by applying the numerically stable covariance factorization techniques developed for Kalman filtering.[5] The factorization methods are all algebraically equivalent to the Kalman formula, but are designed for improved numerical accuracy. Rather than propagating the covariance P, these factorization algorithms recursively compute covariance square roots ($P=SS^T$) or $U-D$ factors ($P=UDU^T$, where U is unit upper triangular and D is diagonal). This approach assures the positivity of P and permits greater precision through improved numerical conditioning.[5]

While several of the factorization filter algorithms are accurate and reliable, only the *UD* formulation rivals the efficiency of the original Kalman formula.[6-8] In fact, for systems with colored process noise and large numbers of bias parameters, the *U-D* method is less costly than the Kalman algorithm.[8] These attributes of the *U-D* filtering technique are utilized in this paper to obtain a numerically improved formulation of the control law and the associated covariance recursions.

Section II of this paper presents the LQG stochastic control problem and classical solution; the *U-D* alternative to this solution is given in Sec. III; and the conclusions are contained in the final section.

II. Problem Definition and Classical Solution

Given the linear dynamic system and observation

$$\left.\begin{aligned} x_{i+1} &= \phi_i x_i + \Gamma_i u_i + G_i \omega_i \quad (1) \\ z_i &= H_i x_i + \nu_i \quad (2) \end{aligned}\right\} i=0,1,\ldots,N-1$$

where $x_i \epsilon R_n$, $u_i \epsilon R_k$, $z_i \epsilon R_m$, and x_0, ω_i, ν_i are zero mean, statistically independent Gaussian variables with covariances

$$E\{\omega_i \omega_j^T\} = Q_i \delta_{ij}, \quad E\{\nu_i \nu_j^T\} = R_i \delta_{ij}, \quad E\{x_o x_o^T\} = \tilde{P}_o,$$

find the control u_i as a function of $z_o,\ldots,z_i$ to minimize the functional

$$J = E\left\{\frac{1}{2} x_N^T \tilde{S}_N x_N + \frac{1}{2} \sum_{i=1}^{N-1} [x_i^T A_i x_i + u_i^T B_i u_i]\right\} \tag{3}$$

where the symmetric arrays $\tilde{S}_N$, A_i, and B_i have the form

$$\tilde{S}_N \geq 0, \quad A_i \geq 0, \quad B_i > 0$$

By properly defining the pairs (Γ_i, u_i) and (G_i, ω_i), we may assume that Q_i and B_i are diagonal. Similarly, an appropriate scaling of the data, Eq. (2), is sufficient to guarantee that each R_i is diagonal.[9] Moreover, since each A_i is nonnegative symmetric, it may be written as

$$A_i = M_i^T \bar{A}_i M_i \tag{4}$$

where

$$\bar{A}_i = \text{diag}\,(\bar{a}_1, \bar{a}_2, \ldots \bar{a}_r) > 0 \quad \text{with} \quad r \leq n \tag{5}$$

This representation of A will be convenient in the ensuing development.

The conventional solution to the preceding problem is[1]:

$$\hat{u}_i = -C_i \hat{x}_i \tag{6}$$

$$C_i = (\Gamma_i^T \tilde{S}_{i+1} \Gamma_i + B_i)^{-1} \Gamma_i^T \tilde{S}_{i+1} \phi_i \tag{7}$$

$$\tilde{S}_i = \phi_i^T \tilde{S}_{i+1} \phi_i - C_i^T \Gamma_i^T \tilde{S}_{i+1} \phi_i + M_i^T \bar{A}_i M_i \tag{8}$$

$\tilde{S}_N$, given

The vector $\hat{x}_i$ represents the minimum variance estimate of x_i given $z_0, z_1, \ldots, z_i$ and may be obtained from the following Kalman filter algorithm.

Measurement update

$$\hat{x}_i = \tilde{x}_i + K_i (z_i - H_i \tilde{x}_i) \tag{9}$$

$$K_i = \tilde{P}_i H_i^T (H_i \tilde{P}_i H_i^T + R_i)^{-1} \tag{10}$$

$$P_i = \tilde{P}_i - K_i H_i \tilde{P}_i \tag{11}$$

Time update

$$\tilde{x}_{i+1} = \phi_i \hat{x}_i + \Gamma_i \hat{u}_i \tag{12}$$

$$\tilde{P}_{i+1} = \phi_i P_i \phi_i^T + G_i Q_i G_i^T \tag{13}$$

Initial conditions

$$\tilde{x}_o = E\{x_o\} = 0 \tag{14}$$

$$\tilde{P}_o, \text{ given} \tag{15}$$

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This filter algorithm is best applied by cycling through Eqs. (9-11) once for each of the m components of z. In this way, the matrix inversion in Eq. (10) is reduced to a simple scalar operation.[9]

The matrices P_i and $\tilde{P}_i$ represent the error covariances of $\hat{x}_i$ and $\tilde{x}_i$, respectively. An important feature of this estimator is the following orthogonality property.[1]

$$E\{(\hat{x}_i - x_i)\hat{x}_i^T\} = 0$$

$$E\{(\tilde{x}_i - x_i)\tilde{x}_i^T\} = 0 \qquad (16)$$

This property permits an easy evaluation of the average behavior of the controlled system. Let

$$X_i = E\{x_i x_i^T\} \qquad (17)$$

$$\bar{X}_i = E\{\hat{x}_i \hat{x}_i^T\} \qquad (18)$$

Then from the orthogonality property we obtain

$$X_i = \bar{X}_i + \tilde{P}_i \qquad (19)$$

$$\bar{X}_{i+i} = (\phi_i - \Gamma_i C_i)(X_i - P_i)(\phi_i - \Gamma_i C_i)^T \qquad (20)$$

The control required at each stage has covariance

$$V_i = E\{\hat{u}_i \hat{u}_i^T\} = C_i(X_i - P_i)C_i^T \qquad (21)$$

Equations (6-15 and 19-21) comprise a concise, easily implemented algorithm, and for this reason they have been widely used. However, experience has shown that the Kalman filter algorithm, Eqs. (9-15), is numerically unstable.[2-4] Severe loss of accuracy may occur during the matrix subtraction, Eq. (11), and subsequent propagation via Eq. (13) can cause further accuracy degradation. These difficulties may not be attributed solely to observability problems, nor can they necessarily be avoided by including high levels of process noise.[4] The root of these numerical problems is the filter algorithm itself – it is simply not a sound computational tool. The control recursion, Eq. (8), is identical in form to the Kalman formula, Eqs. (11) and (13) therefore it too, is a poor computational algorithm. Finally, we note that the covariance recursions, Eqs. (19-21), involve a further unreliable matrix calculation $(X_i - P_i)$, which should be avoided.

Several numerically stable formulations of the Kalman algorithm have been derived.[5,8] These alternative methods achieve significant accuracy improvement by replacing the covariance recursions (11) and (13) with equivalent recursions in judiciously chosen covariance factors. The most efficient factorization method to date involves the covariance decomposition $P = UDU^T$ where U is unit upper triangular and D is diagonal. Time and measurement update algorithms for propagating these U-D factors are derived in Refs. 6-8 and, for completeness, are summarized in Appendix A. Studies have shown the U-D method to be easily implemented, numerically reliable, and computationally efficient.[4,8,14] The following section describes how the U-D algorithms may be used to obtain an accurate and efficient solution to the control recursions, Eqs. (7, 8, and 19-21).

III. *U-D* Factorization of the Control Algorithm

The matrix Riccati equation (8), may be written as follows. Let

$$\tilde{C}_{i+1} \triangleq C_i \phi_i^{-1} \equiv (\Gamma_i^T \tilde{S}_{i+1} \Gamma_i + B_i)^{-1} \Gamma_i^T \tilde{S}_{i+1} \qquad (22)$$

$$S_{i+1} \triangleq \tilde{S}_{i+1} - \tilde{C}_{i+1}^T \Gamma_i^T \tilde{S}_{i+1} \qquad (23)$$

Then $\tilde{S}_i$ is given by

$$\tilde{S}_i = \phi_i^T S_{i+1} \phi_i + M_i^T \bar{A}_i M_i \qquad (24)$$

Note that Eqs. (22-24) are mathematically identical to the Kalman filter equations, Eqs. (10, 11, and 13). Because of duality, the state transition and coefficient matrices for the S recursion are transposed and time is reversed; otherwise, the two recursions are identical. Hence, the U-D filter algorithms can be applied directly to propagate S. To accomplish this we postulate the following hypothetical filtering problem.

Assume process and measurements of the form

$$\left.\begin{aligned} \lambda_i &= \phi_i^T \lambda_{i+1} + M_i^T \xi_i \qquad (25) \\ y_i &= \Gamma_i^T \lambda_{i+1} + \eta_i \qquad (26) \end{aligned}\right\} \; i = N-1, \ldots, 1$$

where $\lambda_i \epsilon R_n$, $y_i \epsilon R_k$, $\xi_i \epsilon R_r$, and ξ_i and η_i are zero mean, purely Gaussian random processes with diagonal covariances $\bar{A}_i$ and B_i, respectively. Further, assume that λ_N is statistically independent of ξ_i and η_i and has mean and covariance

$$E\{\lambda_N\} = 0 \qquad E\{\lambda_N \lambda_N^T\} = \tilde{S}_N$$

The "Kalman filter" for this hypothetical problem is the following. For $i = N-1, \ldots, 0$:

Measurement update

$$\hat{\lambda}_{i+1} = \tilde{\lambda}_{i+1} + \tilde{C}_{i+1}^T (y_i - \Gamma_i^T \tilde{\lambda}_{i+1}) \qquad (27)$$

$$\tilde{C}_{i+1} = (\Gamma_i^T \tilde{S}_{i+1} \Gamma_i + B_i)^{-1} \Gamma_i^T \tilde{S}_{i+1} \qquad (28)$$

$$S_{i+1} = \tilde{S}_{i+1} - \tilde{C}_{i+1}^T \Gamma_i^T \tilde{S}_{i+1} \qquad (29)$$

Time update

$$\tilde{\lambda}_i = \phi_i^T \hat{\lambda}_{i+1} \qquad (30)$$

$$\tilde{S}_i = \phi_i^T S_{i+1} \phi_i + M_i^T \bar{A}_i M_i \qquad (31)$$

$$\tilde{\lambda}_N = 0$$

Thus, the required control gain $C_i = \tilde{C}_{i+1}\phi_i$ is obtained by solving a related Kalman filtering problem. Notice, however, that for statistical guidance analysis [Eqs. (19-21)], it is necessary to obtain the full $k \times n$ array C_i; and hence the gain $\tilde{C}_{i+1}$ is required. But $\tilde{C}_{i+1}$ corresponds to the vector measurement update, Eqs. (27-29). The U-D filter algorithm, which we would like to apply to this problem, assumes scalar measurements and recursively computes k vector gains, $F_1, F_2, \ldots F_k$, instead of the array $\tilde{C}^T$. In Appendix B, we show how the k columns of $\tilde{C}^T$ are linearly related to the vector gains $\{F_i\}_{i=1}^k$. Thus $\tilde{C}$ is obtained by recursively applying the U-D measurement update algorithm followed by a linear transformation of the computed gains. The U-D control recursion is summarized in the following algorithm.

U-D Control Recursion

Let each Γ and B be partitioned such that

$$\Gamma^T = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_k^T \end{bmatrix} \qquad B = \text{diag}\,(b_1, b_2, \ldots, b_k) \qquad (32)\dagger$$

and let $\{\tilde{c}_j\}_{j=1}^k$ represent the k column vectors of $\tilde{C}^T$.

†In order to avoid cumbersome notation, time dependence is included only where necessary to prevent confusion. Thus, in Eqs. (32-35), subscripts are used to denote array components.

Suppose at stage $i+1$ that the $n\times n$ array $\tilde{S}$ is factored such that

$$\tilde{S}=\tilde{U}\tilde{D}\tilde{U}^T \tag{33}$$

The corresponding U-D factors of S and the gain array $\tilde{C}$ [Eqs. (28-29)] may be obtained as follows.

• For $j=1,2,\ldots,\ k$, set $H\equiv\gamma_j^T$, $r\equiv b_j$, $U\equiv U^{(j-1)}$, $D\equiv D^{(j-1)}$, and apply the U-D measurement update of Appendix A to obtain the updated factors $U^{(j)}$, $D^{(j)}$ and the gain $F_j\equiv K$ (Eq. A11). Initially, $U^{(o)}\equiv\tilde{U}$ and $D^{(o)}\equiv\tilde{D}$.

• For $\ell=1,\ldots,j-1$ compute

$$F_\ell:=F_\ell-(\gamma_j^T F_\ell)F_j \tag{34}‡$$

At the conclusion of this recursion, the U-D factors of S and the columns of $\tilde{C}^T$ are given by

$$U=U^{(k)},\quad D=D^{(k)},\quad \tilde{c}_j=F_j,\quad j=1,2,\ldots,k \tag{35}$$

Given the U-D factors of S at stage $i+1$, the $\tilde{U}$-$\tilde{D}$ factors of $\tilde{S}$ at stage i [Eq. (31)] may be obtained by applying the time update algorithm of Appendix A with the W and D arrays set to

$$W\equiv[\phi^T U\quad M^T]\qquad \bar{D}\equiv\text{diag}\,(D,\bar{A}) \tag{36}$$

At each stage, the control gain C_i [Eq. (17)] is given by

$$C_i=\tilde{C}_{i+1}\phi_i \tag{37}$$

The transition matrix required for the controlled state propagation, Eq. (20), may be computed as follows. Let ϕ_j and $\bar{\phi}_j$ denote the jth column vectors of ϕ and $(\phi-\Gamma C)$, respectively. Then

$$\bar{\phi}_j=\phi_j-\sum_{\ell=1}^{k}(c_{\ell j})\gamma_\ell \qquad j=1,\ldots,n \tag{38}$$

Comment

The U-D algorithms permit perfect measurements and semidefinite covariances. Hence, the case where controls are unconstrained ($B_i\equiv 0$) and/or some of the final state errors are unweighted (S_N, singular) can be handled without difficulty.

Note that the backward recursion described here yields the arrays C and $(\phi-\Gamma C)$ at each guidance time. These arrays can be computed and saved for subsequent evaluation of system performance via Eqs. (19-21). As previously noted, the matrix subtraction $(X-P)$ found in Eqs. (20 and 21) should be avoided. Substitution of Eq. (19) into Eqs. (20) and (21), together with an application of Eq. (11), yields the preferred formulas

$$X_i=\bar{X}_i+\tilde{P}_i \tag{39}$$

$$V_i=C_i(\bar{X}_i+Y_i)C_i^T \tag{40}$$

$$\bar{X}_{i+1}=(\phi_i-\Gamma_i C_i)(\bar{X}_i+Y_i)(\phi_i-\Gamma_i C_i)^T \tag{41}$$

where

$$Y_i=\tilde{P}_i-P_i\equiv K_i(H_i\tilde{P}_i H_i^T+R_i)K_i^T \tag{42}$$

and

$$\bar{X}_o=0 \tag{43}$$

The right-hand side of Eq. (42) represents a nonnegative, symmetric array; hence the nonnegative structure of $\bar{X}$ and V is preserved. Notice that when the U-D one-at-a-time measurement update is employed during the forward recursion, Eqs. (9-15), Y_i is given by the summation

$$Y_i=\sum_{j=1}^{m}\alpha^{(j)}K^{(j)}(K^{(j)})^T \tag{44}$$

where $\alpha^{(j)}$ and $K^{(j)}$ represent the innovations covariance and the gain vector for the jth measurement at t_i (see Appendix A).

For the usual occurrence that guidance updates are performed less frequently than measurements are obtained, the Y array is propagated as follows. After a guidance update $Y\equiv 0$ (i.e., $\tilde{P}_i$ replaces P_i) and thereafter at each measurement time $t_{\ell+1}$, the array is updated as

$$Y_{\ell+1}=\phi_\ell Y_\ell\phi_\ell^T+\Big[\sum_{j=1}^{m}\alpha^{(j)}K^{(j)}(K^{(j)})^T\Big]_{\ell+1} \tag{45}$$

until $t_{\ell+1}\equiv t_{i+1}$ the next guidance time.

It is advantageous to apply U-D factorization techniques to the controlled state covariance recursions, Eqs. (41) and (45). [Computations Eqs. (39) and (40), need not be factored, since they are required only for display purposes and do not affect the other recursions.] The U-D update corresponding to Eq. (45) may be accomplished in two steps as follows.

U-D Formulation of the Y Recursion

Given the factorization $Y_\ell=UDU^T$, set

$$W\equiv\phi_\ell U\qquad \bar{D}\equiv D \tag{46}$$

and apply the time update algorithm of Appendix A to obtain the intermediate factors, $\tilde{U}$ and $\tilde{D}$, where

$$\tilde{U}\tilde{D}\tilde{U}^T=\phi_\ell Y_\ell\phi_\ell^T \tag{47}$$

The updated U-D factors of $Y_{\ell+1}$, Eq. (45), may now be computed by repetitive application of the rank-one update that is given in Appendix C; i.e., recursively solve, via the rank-one update formula,

$$U^{(j)}D^{(j)}(U^{(j)})^T=U^{(j-1)}D^{(j-1)}U^{(j-1)^T}+\alpha^{(j)}K^{(j)}K^{(j)}\qquad j=1,\ldots,m \tag{48}$$

with $U^{(o)}=\tilde{U}$ and $D^{(o)}=\tilde{D}$. Upon completion of this recursion, the desired factors of $Y_{\ell+1}$ are $U\equiv U^{(m)}$ and $D\equiv D^{(m)}$.

The factored form of the $\bar{X}$ propagation, Eq. (41), is also accomplished in two steps and involves the same algorithms employed for the Y update.

***U-D* Formulation of the $\bar{X}$ Recursion**

Given the factorizations $\bar{X}=\hat{U}\hat{D}\hat{U}^T$ and $Y=UDU^T$, the updated $\hat{U}-\hat{D}$ factors of $\bar{X}$, Eq. (41) may be obtained as follows. Partition the U and D arrays so that

$$D=\text{diag}\,(d_1,d_2,\ldots,d_n)$$

$$U=[U_1,U_2,\ldots U_n]$$

Compute the U-D factors of $(\bar{X}+Y)$ by applying the rank-one update of Appendix C to recursively solve

$$\hat{U}^{(j)}\hat{D}^{(j)}\hat{U}^{(j)^T}=\hat{U}^{(j-1)}\hat{D}^{(j-1)}\hat{U}^{(j-1)^T}+d_jU_jU_j^T\qquad j=1,\ldots,n \tag{49}$$

with $\hat{U}^{(o)}=\hat{U}$ and $\hat{D}^{(o)}=\hat{D}$. The updated $\hat{U}$-$\hat{D}$ factors of $\bar{X}$ may now be computed by applying the time update algorithm

‡The symbol ": =" denotes replacement in computer storage.

of Appendix A with

$$W \equiv (\phi - \Gamma C)\hat{U}^{(n)} \quad \text{and} \quad \tilde{D} \equiv \hat{D}^{(n)} \tag{50}$$

Remarks

1) The recursion [Eq. (49)] may involve considerably less than the n steps indicated, since Y can be rank-deficient. In this case, some of the d_j terms will be identically zero, and the rank-one update corresponding to those terms may be omitted.

2) After completing the factorization of $\bar{X}+Y$ via Eq. (49), the control covariance [Eq. (40)] may be computed as follows.

Let

$$\bar{C} = C\hat{U}^{(n)} \qquad \bar{D} = \hat{D}^{(n)} \tag{51}$$

Then

$$V = \bar{C}\bar{D}\bar{C}^T$$

IV. Summary

The U-D factorization technique has been applied to obtain a numerically stable formulation of the LQG stochastic control law. The backward recursion for computing optimal control gains was solved by a direct application of the U-D filter algorithms coupled with an appropriate linear transformation. Rearrangement of the controlled state covariance equations eliminated a numerically hazardous matrix subtraction and provided a stable and efficient U-D formula for performing statistical guidance analysis.

From an applications point of view, it is satisfying to note that the complete problem, including both estimation and control, can be solved with only three U-D algorithms: the weighted Gram-Schmidt time update, the measurement update, and the rank-one update formulas. These algorithms have been tested on a variety of problems and have proven to be reliable, efficient, and easy to implement.[4,13] Experience has shown that when the U-D algorithms are applied so as to maximize efficiency, taking advantage of special system structure to minimize storage and computation, the resulting implementations rival the efficiency of conventional covariance methods. Moreover, for systems with colored noise and large number of bias parameters, the U-D method can be less costly than even the conventional covariance formula.[8,14]

Appendix A: U-D Filter Algorithms

Suppose the n-dimensional error covariance matrix, P, is factored such that

$$P = UDU^T \tag{A1}$$

where U is upper triangular with unit diagonals and $D = \text{diag}(d_1,\ldots,d_n)$. The matrices U and D are referred to as the U-D factors of P. They are unique, provided that P is positive definite and can be constructed using a Cholesky factorization.[8]

Algorithms are presented here for performing measurement and time updating of the U-D factors. These algorithms correspond to the conventional Kalman formulas, Eqs. (10, 11, and 13).

U-D Measurement Update Algorithm

Given a priori covariance factors $\tilde{U}$ and $\tilde{D}$ and a scalar measurement $z = Hx + \nu$, where $E(\nu^2) = r$, Bierman[6] has shown that the updated U-D covariance factors and the Kalman gain ($U, D,$ and K, respectively) can be obtained as follows:

$$f^T = H\tilde{U} \qquad f^T = (f_1,\ldots,f_n) \tag{A2}$$

$$v = \tilde{D}f \qquad v_j = \tilde{d}_j f_j \tag{A3}$$

$$\bar{K}_1^T = (v_1, \overbrace{0,\ldots,0}^{n-1}) \tag{A4}$$

$$\alpha_1 = r + v_1 f_1 \tag{A5}$$

If $\alpha_1 = 0$, omit Eq. (A6)§

$$d_1 = (r/\alpha_1)\tilde{d}_1 \tag{A6}$$

For $j = 2,\ldots,n$ cycle through Eqs. (A7-A11)

$$\alpha_j = \alpha_{j-1} + v_j f_j \tag{A7}$$

If $\alpha_j = 0$, omit Eqs. (A8-A11)§

$$d_j = (\alpha_{j-1}/\alpha_j)\tilde{d}_j \tag{A8}$$

If $d_j = 0$, skip to Eq. (A11)§

$$\lambda_j = f_j/\alpha_{j-1} \tag{A9}$$

$$U_j = \tilde{U}_j + \lambda_j \bar{K}_{j-1} \tag{A10}$$

$$\bar{K}_j = \bar{K}_{j-1} + v_j \tilde{U}_j \tag{A11}$$

where $U = [U_1, U_2 \ldots, U_n]$. The component U vectors have the form

$$U_j^T = [U_j(1), \ldots U_j(j-1), 1, 0, \ldots, 0]$$

and $D = \text{diag}(d_1,\ldots,d_n)$. The Kalman gain is given by

$$K = \bar{K}_n/\alpha_n \tag{A12}$$

where α_n is the innovations covariance.

Modified Gram-Schmidt techniques may be used to accomplish time updating of the U-D factors[7,8] and the resulting algorithm is the following.

Modified Gram-Schmidt Time Update Algorithm

Let

$$W = [\overbrace{\phi U}^{n} \ \overbrace{G}^{p}] \qquad \text{with row vectors } \{w_i\}_{i=1}^n$$

$$\bar{D} = \text{diag}(\overbrace{D,}^{n} \ \overbrace{Q}^{p}) = \text{diag}(\bar{d}_1, \bar{d}_2, \ldots \bar{d}_{n+p})$$

The $\tilde{U}$-$\tilde{D}$ factors of $\tilde{P} = W\bar{D}W^T$ [Eq. (13)] may be computed as follows.

For $j = n, n-1, \ldots, 1$, evaluate Eqs. (A13-A15).

$$\tilde{d}_j = w_j^T \bar{D} w_j \tag{A13}$$

If $\tilde{d}_j = 0$, omit Eqs. (A14) and (A15).

$$\left.\begin{aligned} \tilde{u}_{ij} &= \frac{1}{\tilde{d}_j}[w_i^T \bar{D} w_j] && \text{(A14)} \\ w_i &:= w_i - \tilde{u}_{ij} w_j && \text{(A15)} \end{aligned}\right\} i = 1, 2, \ldots, j-1$$

Propagation of U-D factors can also be accomplished by applying modified Givens techniques.[8] However, the modified Givens method requires more equations to describe it and, except in unusual circumstances, offers only modest computational savings over the Modified Gram-Schmidt algorithm.

§These steps are exercised when perfect measurements and/or semidefinite covariances are processed.

Appendix B: Linear Filter Gain Relationships

The following lemma, suggested by G.J. Bierman, relates recursive filtering gains to gain arrays obtained by collective data processing.

Lemma

Let the jth component of a vector measurement $y=\Gamma^T\lambda+\eta$ be given by

$$y_j=\gamma_j^T\lambda+\eta_j \qquad j=1,\ldots,k \tag{B1}$$

and define the recursion

$$\hat{\lambda}^{(j)}=\hat{\lambda}^{(j-1)}+F_j(y_j-\gamma_j^T\hat{\lambda}^{(j-1)}) \tag{B2}$$

with $\hat{\lambda}^{(o)}$ given.

Further assume that $\hat{\lambda}^{(k)}=\hat{\lambda}$, where $\hat{\lambda}$ is obtained by the vector measurement update

$$\hat{\lambda}=\hat{\lambda}^{(o)}+\bar{C}^T(y-\Gamma^T\hat{\lambda}^{(o)}) \qquad \bar{C}^T=[\bar{c}_1\bar{c}_2\ldots\bar{c}_k] \tag{B3}$$

Then, the vector $\{\bar{c}_j\}$ may be obtained from the set $\{F_j\}$ via the following recursion. Let

$$F_1^{(1)}=F_1 \tag{B4}$$

For $j=2,\ldots,k$ cycle through Eqs. (B5 and B6)

$$F_j^{(j)}=F_j \tag{B5}$$

$$F_i^{(j)}=F_i^{(j-1)}-(\gamma_j^TF_i^{(j-1)})F_j^{(j)} \qquad i=1,2,\ldots,j-1 \tag{B6}$$

Upon completion of this recursion

$$\bar{c}_j\equiv F_j^{(k)} \tag{B7}$$

Appendix C: Rank-One Matrix Factorization

Let the symmetric matrix $\bar{P}$ be factored such that $\bar{P}=\bar{U}\bar{D}\bar{U}^T$ [Eq. (A1)]. Then, the corresponding factors of

$$P=\bar{P}+c\lambda\lambda^T \tag{C1}$$

for $c>0$ and vector quantity λ may be obtained as follows.

Evaluate Eqs. (C2-C10) recursively for $j=n,n-1,\ldots,2$.

$$\alpha_j=c_j\lambda_j \qquad (c_n=c) \tag{C2}$$

If $(\alpha_j=0)$, omit Eqs. (C3-C10).

$$d_j=\bar{d}_j+\alpha_j\lambda_j \tag{C3}$$

$$v_j=\alpha_j/d_j \tag{C4}$$

$$\beta_j=\bar{d}_j/d_j \tag{C5}$$

$$\left.\begin{aligned} f_i &:=\lambda_i && \text{(C6)}\\ \lambda_i &:=\lambda_i-\lambda_j\bar{u}_{ij} && \text{(C7)}\\ u_{ij} &=\bar{u}_{ij}+\lambda_iv_j, \text{ for } \beta_j>0.1 && \text{(C8)}\\ u_{ij} &=\beta_j\bar{u}_{ij}+f_iv_j, \text{ for } \beta_j\le 0.1 && \text{(C9)}\end{aligned}\right\} i=1,2,\ldots,j-1$$

$$c_{j-1}=\beta_jc_j \tag{C10}$$

$$d_1=\bar{d}_1+c_1\lambda_1^2 \tag{C11}$$

Equations (C2-C11) represent a modest rearrangement of the Agee-Turner rank-one update,[10] coupled with an alternative calculation, Eq. (C9), which our experience has shown should be included for greater numerical stability.[11,12]

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